PHYSICAL REVIEW VOLUME 185, NUMBER 5 25 SEPTEMBER 1969

Low-Energy Pion-Pion Scattering in the Unitarized Strip Approximation

R. C. JOHNSON*†
Department of Physics, University of Toronto, Toronto 181, Canada

AND

P. D. B. COLLINS
Department of Physics, Durham University, Durham City, England

(Received 23 May 1969)

We present an improved set of parameters for a successful bootstrap of the ρ and Pomeranchon Regge trajectories in pion-pion scattering. The resulting π-π amplitudes are compared with the experimental data, with particular reference to the low-energy S-wave phase shifts, which agree fairly well with present indications. We obtain for the trajectory intercepts $\alpha_P(0)=1.0$ and $\alpha_\rho(0)=0.55$; for the ρ-meson width $\Gamma_\rho=143$ MeV; and for the S-wave scattering lengths $a_0=(0.15\pm0.05)m_\pi^{-1}$, $a_2=(-0.04\pm0.015)m_\pi^{-1}$.

INTRODUCTION

IN a recent paper[1] an improved form of the strip approximation[2] was proposed as a scheme for the calculation of hadronic four-line connected parts in accordance with the postulates of maximal analyticity of the first and second kinds.[3] An analytic and crossing-symmetric amplitude is constructed by adding to the strips of double spectral function containing the leading Regge poles in each channel[4] elastic-unitary double spectral functions calculated using the Mandelstam iteration method.[5] Unitarity is then imposed on the partial-wave amplitudes through the Frye-Warnock[6] N/D equations, with inelasticity obtained from the double spectral function.

The scheme was first applied[1] to a ρ-meson bootstrap in π-π scattering, and has since been used[7] to bootstrap both ρ and P trajectories in this process. The purpose of this note is to report a more refined set of parameters for this problem and to present and discuss the corresponding π-π S-wave phase shifts. These have over-all features (sign and magnitude) of the sort indicated by recent experimental and theoretical analyses.

SELF-CONSISTENT CALCULATIONS

Full accounts of the calculational procedure have already been given[1,7]; they will not be repeated. Briefly, crossed-channel Regge poles are parametrized in terms of dispersion relations for their residue and trajectory functions, and the parameters are varied to achieve coincidence with the output of the N/D equations below threshold. The shape of the direct-channel output trajectories demands that the input trajectory functions have large imaginary parts at fairly low energies. As described in Ref. 7, the way to eliminate the resulting peaks in the t-channel cross section (which has no s-channel counterparts) is to demand that the real parts of the input residue functions be very small where the imaginary parts of the input trajectories have their maxima.

Imposing self-consistency both below and above threshold in this way leads[7] to a fairly narrow range of acceptable solutions for the ρ and P, with trajectory intercepts $1.2\geqslant\alpha_P(0)\geqslant0.89$ and $0.69\geqslant\alpha_\rho(0)\geqslant0.32$. The corresponding ρ-meson widths at the upper and lower limits, respectively, are $\Gamma_{\rm in}=113$ MeV, $\Gamma_{\rm out}=102$ MeV, and $\Gamma_{\rm in}=182$ MeV, $\Gamma_{\rm out}=218$ MeV. Previously we reported[7] a solution with trajectory intercepts $\alpha_P(0)=1.0$, $\alpha_\rho(0)=0.55$, and input and output ρ widths of $\Gamma_{\rm in}=135$ MeV, $\Gamma_{\rm out}=155$ MeV. Now we have improved this slightly and have found a set of parameters[8] that give the same trajectory intercepts as above, but identical direct- and crossed-channel ρ widths of 143 MeV. This compares very favorably with nearly all earlier results,[9] and in view of the uncertainties in the measured ρ width,[10] it is completely satisfactory numerical perdiction.

Our value for the Pomeranchon residue $\gamma_P(0)$ corresponds to an asymptotic π-π total cross section of 26 mb, compared with an estimate[11] from factorization of 10–14 mb. We have not been able to improve agreement while maintaining the bootstrap by, for instance, separating the vacuum Regge-pole exchange into P plus P'

* Supported by the National Research Council of Canada.

† Present address: Mathematics Department, Durham University, England.

[1] P. D. B. Collins and R. C. Johnson, Phys. Rev. **177**, 2472 (1969).

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).

[3] G. F. Chew, *The Analytic S-Matrix* (W. A. Benjamin, Inc., New York, 1966).

[4] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

[5] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[6] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1961).

[7] P. D. B. Collins and R. C. Johnson, Phys. Rev. **182**, 1755 (1969).

[8] These are, in the notation of Ref. 7, $t_B=8.2m_\pi^2$, $c_1=1.38\times10^{-3}m_\pi^{-2}$, $t_\alpha=133.3m_\pi^2$, $\lambda=0.1$, $\Delta=5m_\pi^2$, $s_1=1000m_\pi^2$, $d=208m_\pi^2$, $a_2=208m_\pi^2$, $b_2=188m_\pi^2$, $\alpha_\rho(\infty)=-0.35$, $\alpha_P(\infty)=0.15$, $c_2^\rho=12.1$, and $c_2^P=8.45$. Input and output trajectories differ very little from those given in Fig. 4 of Ref. 7, and direct- and crossed-channel $I=l=1$ cross sections show symmetrical peaks identical to those shown in Fig. 6 of that paper.

[9] For a review and references see P. D. B. Collins and E. J. Squires, *Regge Poles in Particle Physics* (Springer-Verlag, Berlin, 1968).

[10] Particle Data Group, Rev. Mod. Phys. **41**, 109 (1969).

[11] This estimate was made by fitting the total cross section data for πN and NN scattering given by W. Galbraith *et al.* [Phys. Rev. **138**, B913 (1965)] with three-parameter formulas $\sigma=A+Bs^\alpha$, using $A_{\pi\pi}A_{NN}=A_{\pi N}^2$.

contributions, because these calculations show no sign of producing the output P' trajectory necessary for self-consistency.[12] Our inability to include this presumably important feature of the π-π amplitude must be borne in mind when assessing the other secondary properties of the bootstrap solutions.

We quote our prediction for the S-wave phase shifts because these quantities are of considerable current interest. However, we must point out that there are three reasons why this calculation may not give completely satisfactory results. The first is the problem,[3] common to all approaches relying on partial-wave dispersion relations, that the S waves are sensitive to distant singularities. Physically these correspond to short-range forces, and there is no angular momentum barrier to damp their effects. Although part of the distant singularities are included in the strip approximation,[1–4] we are explicitly neglecting the innermost pieces of the double spectral functions (where s and t are both large) which may be important for the core of the interaction.

The second problem is more technical; it is the fact that in (for example) the isospin-zero channel for $l \lesssim 0.33$ there is a violation of unitarity[7] near the upper boundary of the s-channel strip due to the large size of the double spectral function.[13] This renders strictly insoluble (by our matrix-inversion method[14]) the integral equation for the N function. However, we are encouraged to believe this to be of no practical importance (at any rate, as far as low-energy phase shifts are concerned) because the numerical procedure for solving the N/D equations shows no sign of upset near the relevant angular momenta and provides a smooth and satisfactory continuation of the solutions to lower l values.[15]

Perhaps the most serious difficulty is the neglect of inelastic channels other than those involving pions. The π-π system is coupled to $K\bar{K}$, $N\bar{N}$, etc., and, even though their thresholds are comparatively high, it is possible that these channels affect the S wave significantly. For instance, in S-wave potential scattering it is much easier to produce bound states than resonances, so that if dipion $I=J=0$ resonances exist,[10] they are likely to be bound states of higher threshold channels, and to appear as CDD poles[16] in π-π scattering.[17]

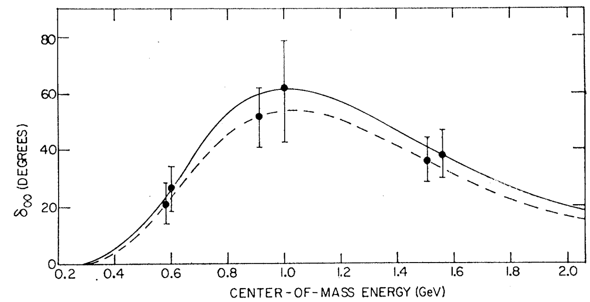

FIG. 1. The isospin-zero S-wave phase shift δ_{00} as a function of c.m. energy in the direct channel (full line) and in the crossed channel (dashed line). The significance of the error bars is described in the text.

S-WAVE PHASE SHIFTS

Bearing these matters in mind, we can easily project out the $l=0$ states in both s and t channels from the self-consistent solutions.

There is significant scattering in the isospin-zero S wave, and Fig. 1 shows the phase shift δ_{00} for c.m. energies up to 2.0 GeV, for the set of parameters[8] that correspond to the bootstrapped ρ of 143-MeV width. In both direct and crossed channels δ_{00} is positive, rising to a substantial maximum in the region of 1 GeV. The error bars in the figure represent the range of variation of δ_{00} with the variation of the acceptable bootstrap solution. The upper (lower) limits for both channels correspond to the upper (lower) limit of self-consistent trajectory intercepts.

Figure 2 is the corresponding Argand diagram for the $I=0$ S wave, showing the violation of unitarity that occurs at about 3.9 GeV, where the inelasticity function η vanishes.

Our direct-channel value for the $I=0$ scattering length is

$$a_0=(0.15\pm0.05)m_\pi^{-1}\,, \tag{1}$$

where the quoted error reflects the range of bootstrap results. The crossed-channel value is some 10% smaller, since the S waves are not perfectly self-consistent.

Figure 3 depicts the isospin-two-phase shift δ_{20}, which is small and negative in each channel. The error bars have the same significance as in Fig. 1, and we find for

[12] Also there is no sign that secondary ρ-like trajectories can be successfully included in this bootstrap.

[13] In the isospin $I=1$ channel, where the potential is smaller, the critical l value is 0.3; in the $I=2$ channel (where the potential is negative), there is no violation above $l=-0.5$.

[14] C. E. Jones and G. Tiktopoulos, J. Math. Phys. **7**, 311 (1966).

[15] We have used the N/D equations without problems down to $l=-0.35$, following the ρ trajectory down to its end point. The fact that the output ρ and P trajectories remain smooth and essentially parallel (Ref. 7), lying on top of their t-channel counterparts over the whole of the energy range investigated, confirms our viewpoint. The same conclusion was reached in previous work [P. D. B. Collins, Phys. Rev. **142**, 1163 (1966)] in a calculation where the violation was more severe.

[16] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[17] Studies of low-energy nucleon-nucleon scattering [for a review see C. Lovelace, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (North-Holland Publishing Co., Amsterdam, 1968)] suggest that the π-π S-wave interaction is strongly coupled to the $N\bar{N}$ channel. Also we note that omission of the $K\bar{K}$ channel probably explains why these calculations cannot produce a trajectory for the $f'(1515)$ [which decays much more strongly to $K\bar{K}$ than to $\pi\pi$ (Ref. 10)]. Should the f' turn out after all to be the P' recurrence, then the strong binding in the $K\bar{K}$ channel necessary to produce this high-lying trajectory would almost certainly be reflected in the π-π S-wave phase shifts. Also we note evidence (Ref. 10) for a $I^GJ^P=0^+0^+$ state near 1070 MeV which couples mainly to $K\bar{K}$, but which also should appear in the $\pi\pi$ channel.

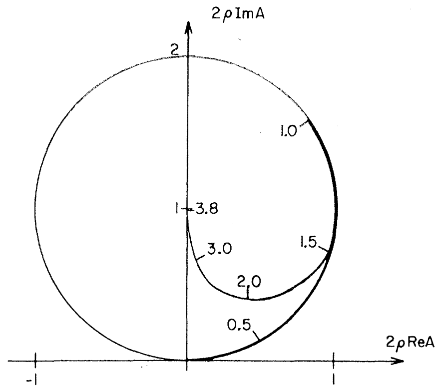

FIG. 2. The Argand diagram (Lovelace, Ref. 23) for the isospin-zero S-wave amplitude A, where ρ is the phase-space factor of Refs. 1 and 7. Center-of-mass energies are marked in GeV units. The inelasticity factor η is essentially unity below about 1.5 GeV, and the violation of unitarity ($\eta=0$) occurs at about 3.9 GeV [$s_1=1000m_\pi^2=(4.43\text{ GeV})^2$].

the s-channel scattering length

$$a_2=(-0.04\pm0.015)m_\pi^{-1}, \qquad (2)$$

and an 8–10% smaller value in the t channel.

It is not difficult to understand qualitatively how our calculations produce these results. Firstly, in the direct channels with $I=0$ and $I=1$ the forces between the scattering pions come from attractive $I=0$ and $I=1$ exchanges. These reinforce strongly in $I=0$ to give the high-lying P trajectory plus the considerable S-wave interaction, and reinforce to a lesser extent in $I=1$, giving the ρ. In $I=2$, the crossing matrix[18] results in a repulsive force from $I=1$ exchange, and this is sufficient to overcome the attractive isoscalar exchange. Solutions with higher-lying (lower-lying) ρ and P trajectories have stronger (weaker) potentials, giving larger (smaller) S-wave interactions in both channels. Isovector exchanges vary more with trajectory height.

The contribution of the asymptotic strip of double spectral function corresponding to the exchange of the t-channel Pomeranchon is repulsive,[19] and in earlier calculations[20] there was the difficulty that this implied a negative absorptive part in the t channel, in violation of unitarity for $4m_\pi^2<t<16m_\pi^2$. The inclusion of the corner double spectral functions has corrected this, producing a positive imaginary part and a perfectly sensible behavior for the S-wave phase shifts. Since we have imposed unitarity explicitly, this is just what one would expect.

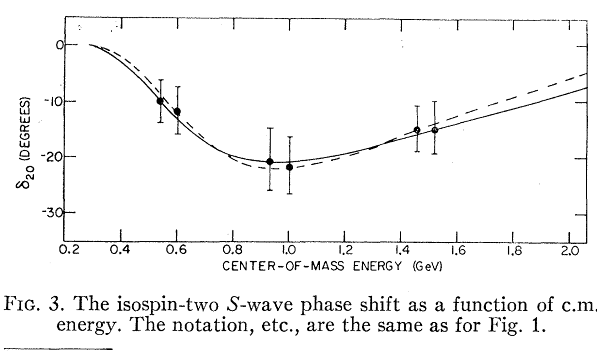

FIG. 3. The isospin-two S-wave phase shift as a function of c.m. energy. The notation, etc., are the same as for Fig. 1.

[18] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[19] G. F. Chew, Phys. Rev. **140**, B1427 (1965).
[20] P. D. B. Collins, Phys. Rev. **142**, 1163 (1966).

It is gratifying that all the acceptable self-consistent solutions show almost equal direct- and crossed-channel S waves, and that the over-all nature of the phase shifts changes rather little with the intercepts of the boot-strapped trajectories. We have achieved approximately self-consistent S waves without explicitly requiring them. Probably this is the best one can do in this sort of calculation, although *a priori* one might have hoped that the requirement of identical s- and t-channel S waves would help to narrow the acceptable range of bootstrap solutions.

DISCUSSION

A variety of sources tend to support the main features of our S-wave amplitudes.

Experimental information on π-π scattering has been obtained from an examination of related processes, and the results are rather model-dependent and subject to large uncertainties. However, several analyses of peripheral pion production[21] concur in suggesting that in the ρ region δ_{00} is large and positive while δ_{20} is small and negative, in general agreement with Figs. 1–3. Some solutions[21] for δ_{00} increase through $\frac{1}{2}\pi$ near the ρ, indicating the presence of a resonance (which we shall denote σ), and an analysis[22] of $p\bar{p}\rightarrow 3\pi$ data supports this possibility and agrees with earlier work[23] on πN backward dispersion relations.

Recent theoretical ideas[24] favor[25] the existence of a σ close to the ρ in mass, and its presence is required by at least one[26] dynamical calculation. However, many approaches[27] merely demonstrate that a σ is a reasonable possibility, and they cannot distinguish between a

[21] P. B. Johnson *et al.*, Phys. Rev. **176**, 1651 (1968); L. J. Gutay *et al.*, Purdue Report No. COO-1428-65, 1968 (unpublished); J. P. Baton and G. Laurens, Phys. Rev. **176**, 1574 (1968); E. Malamud and P. E. Schlein, Phys. Rev. Letters **19**, 1056 (1967); W. D. Walker *et al.*, *ibid.* **18**, 630 (1967); S. Marateck *et al.*, *ibid.* **21**, 1613 (1968); I. F. Corbett *et al.*, Phys. Rev. **156**, 1451 (1967); A. B. Clegg, *ibid.* **163**, 1664 (1967); Nucl. Phys. **B6**, 75 (1968).
[22] J. F. L. Hopkinson and R. G. Roberts, Nuovo Cimento **59A**, 181 (1969).
[23] C. Lovelace, R. M. Heinz, and A. Donnachie, Phys. Letters **22**, 332 (1966); C. Lovelace, CERN Report No. Th. 838, 1968 (unpublished).
[24] G. Veneziano, MIT Report, 1969 (unpublished).
[25] V. Barger and D. Cline, Phys. Rev. (to be published).
[26] R. C. Johnson, Phys. Rev. Letters **22**, 1143 (1969).
[27] R. Arnowitt *et al.*, Phys. Rev. **175**, 1820 (1968); E. P. Tryon, Phys. Rev. Letters **20**, 769 (1968); L. S. Brown and R. L. Goble, *ibid.* **20**, 346 (1968); G. Auberson, O. Piguet, and G. Wanders, Phys. Letters **28B**, 41 (1968). The necessity for a σ meson for the saturation of the π-π Adler sum rule [S. L. Adler, Phys. Rev. **140**, B736 (1965)] has been removed by recent discussions of high-energy contributions [R. C. Johnson, Nuovo Cimento Letters **2**, 50 (1969); K. V. Vasavada, Phys. Rev. **178**, 2350 (1969); *ibid.* (to be published)].

resonance and a large nonresonant δ_{00}. There is general agreement that δ_{20} is small and negative.

The majority of evidence[21–28] suggests that the scattering lengths a_0 and a_2 are, respectively, positive and negative, with $|a_2|<|a_0|$, as expected from the phase shifts at higher energies.[21] The only outstanding disagreement is with a model analysis[29] of threshold pion production (which is supported by investigations[30] of πN partial-wave dispersion relations) that suggests a_0 negative, and δ_{00} changing sign quickly to become large and positive. We note that this seems to conflict with current algebra[28,31] and other[32] treatments of the same process, but it is supported by work of Pišút.[33]

However, Pišút's favored solution[33] predicts $2a_0-5a_2<0$, in contradiction to the suggestion[34] from dispersion sum rules that if the $I=2$ π-π interaction is not large, then $2a_0-5a_2$ is positive. In the absence of an important σ meson, one finds[34] $2a_0-5a_2\approx0.5m_\pi^{-1}$, and Eqs. (1) and (2) are in good agreement with this figure, which is close to the current algebra value[28] of $0.54m_\pi^{-1}$ (corresponding to the measured[10] charged-pion decay rate).

The exact nature of π-π S-wave scattering near threshold will probably be determined most easily from high-statistics experiments on K_{l4} decay.[35,36] The present limited data seem to favor[36] a rather larger scattering length than ours [$a_0=(0.7\pm0.37)m_\pi^{-1}$], although a negative value is not ruled out. There is evidence[37] from the nonleptonic decays $K_1{}^0\to\pi\pi$ that at the kaon mass (498 MeV) $|\delta_{00}-\delta_{20}|\approx40°$, in moderately good agreement with our results.

Note that had these calculations produced a P' trajectory or a steeper P cutting $\alpha_P(t)=0$ for some $t<0$, then Levinson's theorem would suggest[38] $a_0<0$. Our failure to produce the P', and the evidence that $a_0>0$ suggests that the main dynamics of this trajectory (and hence of its first recurrence[39]) lies in other channels, and that it is a CDD pole[16] in π-π scattering.

CONCLUSIONS

We find perfectly sensible ρ and P trajectories, with parameters quite close to those determined phenomenologically[9] for small $|t|$, as well as very reasonable phase shifts for the low partial waves[40] up to at least 1 GeV. This adds to our confidence that the scheme takes adequate account of the short-range forces arising from higher Born approximations to the left-hand cut.[41] Since Regge-pole asymptotic behavior, crossing symmetry, and low-energy unitarity are required of the model, our amplitude has many of the features likely for the real physical amplitude for a wide range of s and t.

Its deficiencies are the exclusion of Regge cuts, which are now under intensive investigation both theoretically[42] and phenomenologically,[43] and the fact that our trajectories behave for large $|t|$ very differently from what experiment seems to indicate.[9,44] Regge-pole phenomenology is consistent with most trajectories being steep and more or less linear with t, and we have pointed out previously[7,45] that the narrowness of observed recurrences makes such a behavior incomprehensible from the viewpoint of this sort of dynamics.[46] It is not obvious that the multiperipheral calculations[42,47] will be any better in this respect.

The fact that our results are impressively successful in some respects and completely deficient in others makes their precise significance difficult to assess. But this does seem to be the most successful bootstrap calculation to date; it must encourage us to reconsider the problems raised in Refs. 7 and 45.

[28] S. Weinberg, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[29] S. Humble and T. D. Spearman, Phys. Rev. **171**, 1724 (1968); S. Humble, Nucl. Phys. **B8**, 695 (1968).

[30] G. W. Ellison and S. Humble, Phys. Rev. **173**, 1563 (1963).

[31] M. G. Olsson and L. Turner, Phys. Rev. Letters **20**, 1127 (1968); Phys. Rev. (to be published); R. G. Levers, Ph.D. thesis, University of Alberta, 1969 (unpublished).

[32] R. G. Roberts and F. Wagner, CERN Report No. Th. 1014, 1969 (unpublished).

[33] J. Pišút, Nucl. Phys. **B8**, 159 (1968).

[34] R. C. Johnson (unpublished).

[35] A. Pais and S. B. Treiman, Phys. Rev. **168**, 1858 (1968).

[36] F. A. Berends, A. Donnachie, and G. C. Oades, Phys. Rev. **171**, 1457 (1968).

[37] B. Gobbi *et al.*, Phys. Rev. Letters **22**, 682 (1969); **22**, 865(E) (1969).

[38] G. F. Chew, Phys. Rev. Letters **16**, 60 (1966); P. D. B. Collins, Phys. Rev. **142**, 1156 (1966).

[39] Most likely the $f(1264)$, but possibly the $f'(1515)$.

[40] Detailed examination of our $I=l=1$ phase shifts indicates a P-wave scattering length $a_1=(0.04\pm0.01)m_\pi^{-3}$, in excellent agreement with current algebra estimates [see, e.g., Ref. 28 and M. G. Olsson, University of Wisconsin Report No. COO 228, 1969 (unpublished)], dispersion theory predictions [M. G. Olsson, Phys. Rev. **162**, 1338 (1967)], and experimental indications [J. Pišút and M. Roos, Nucl. Phys. **B6**, 325 (1968)].

[41] P. D. B. Collins and R. C. Johnson, Phys. Rev. **169**, 1222 (1968).

[42] W. R. Frazer, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[43] F. Henyey *et al.*, Phys. Rev. Letters **21**, 946 (1968); Phys. Rev. (to be published); R. C. Arnold, *ibid.* **153**, 1523 (1967); Argonne National Laboratory Report No. ANL/HEP 6804 (unpublished); J. Finkelstein and M. Jacob, Nuovo Cimento **56A**, 681 (1968).

[44] A very recent review is given by G. E. Hite, University of Oregon Report, 1969 (unpublished). See also Ref. 9.

[45] P. D. B. Collins, R. C. Johnson, and E. J. Squires, Phys. Letters **26B**, 223 (1968).

[46] As in potential scattering, $\mathrm{Im}\alpha(t)$ (which is proportional to the total widths of the recurrences) necessarily becomes large not far above threshold (Refs. 1 and 7).

[47] G. F. Chew, Phys. Rev. Letters **22**, 364 (1969).